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On the Convergence of Polynomial Approximation of Rational Functions

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This paper investigates the convergence condition for the polynomial approximation of rational functions and rational curves. The main result, based on a hybrid expression of rational functions (or curves), is that two-point Hermite interpolation converges if all eigenvalue moduli of a certain $r \times r$ matrix are less than 2, where r is the degree of the rational function (or curve), and where the elements of the matrix are expressions involving only the denominator polynomial coefficients (weights) of the rational function (or curve). As a corollary for the special case of $r = 1$, a necessary and sufficient condition for convergence is also obtained which only involves the roots of the denominator of the rational function and which is shown to be superior to the condition obtained by the traditional remainder theory for polynomial interpolation. For the low degree cases ($r = 1$, 2, and 3), concrete conditions are derived. Application to rational Bernstein–Bézier curves is discussed.

1. INTRODUCTION

A degree r *rational Bernstein polynomial* is given by the equation

$$R(t) = \frac{P(t)}{w(t)} = \frac{\sum_{i=0}^{r} w_i R_i B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)}, \qquad 0 \leqslant t \leqslant 1, \tag{1}$$

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where $R_i \in \mathscr{R}$ are real numbers, $w_i \in \mathscr{R}$ are the weights, and $B_i^r(t) = \binom{r}{i} t^i (1-t)^{r-i}$ denote the *Bernstein basis polynomials*. Throughout this paper, we will use terms *rational Bernstein polynomial* and *rational function* interchangeably. Although we assume R_i and w_i are real, the general results also hold for complex numbers.

Any degree r *Bernstein polynomial* $w(t)$ can be expressed exactly as a degree $r+1$ Bernstein polynomial,

$$w(t) = \sum_{i=0}^{r+1} \hat{w}_i B_i^{r+1}(t), \tag{2}$$

where

$$\hat{w}_i = \left(1 - \frac{i}{r+1}\right) w_i + \frac{i}{r+1} w_{i-1}. \tag{3}$$

This is called *degree elevation* (see [7] for details).

Denote by

$$h^{m,n}(t) = \sum_{i=0}^{m+n-1} h_i^{m,n} B_i^{m+n-1}(t) \tag{4}$$

the degree $m+n-1$ Bernstein polynomial which satisfies

$$\begin{aligned} \frac{d^j h^{m,n}(0)}{dt^j} &= \frac{d^j R(0)}{dt^j}, \qquad j = 0, 1, ..., m-1, \\ \frac{d^j h^{m,n}(1)}{dt^j} &= \frac{d^j R(1)}{dt^j}, \qquad j = 0, 1, ..., n-1. \end{aligned} \tag{5}$$

We will say that $h^{m,n}(t)$ is the *$h\langle m, n\rangle$ polynomial approximation* to $R(t)$. The approximation domain is $0 \leqslant t \leqslant 1$ unless stated otherwise.

In [5], another polynomial approximation technique, using so-called *hybrid polynomials*, is introduced. A hybrid polynomial $\tilde{H}^{m,n}(t)$ is a degree $m+n$ Bernstein polynomial with one coefficient $V^{m,n}(t)$ being a degree r rational function. The hybrid polynomial is equivalent to the original rational function $R(t)$,

$$\tilde{H}^{m,n}(t) \equiv R(t) = \sum_{i=0,\, i \neq m}^{m+n} H_i^{m,n} B_i^{m+n}(t) + V^{m,n}(t)\, B_m^{m+n}(t), \quad 0 \leqslant t \leqslant 1, \tag{6}$$

where

$$V^{m,n}(t) = \frac{\sum_{i=0}^{r} w_i V_i^{m,n} B_i^r(t)}{\sum_{i=0}^{r} w_i B_i^r(t)}. \tag{7}$$

Hybrid polynomials can be used to compute a polynomial approximation to rational functions. If we replace $V^{m,n}(t)$ with a constant $H_m^{m,n}$ such that $\mathrm{Min}_{0 \leqslant i \leqslant r} V_i^{m,n} \leqslant H_m^{m,n} \leqslant \mathrm{Max}_{0 \leqslant i \leqslant r} V_i^{m,n}$, then the Bernstein polynomial

$$H^{m,n}(t) = \sum_{i=0}^{m+n} H_i^{m,n} B_i^{m+n}(t) \tag{8}$$

gives a polynomial approximation to rational function $R(t)$. We call $H^{m,n}(t)$ the $H\langle m, n\rangle$ polynomial approximation to $R(t)$. An advantage in applying hybrid polynomials to polynomial approximation is that error bounding becomes trivial. For example, if we choose $H_m^{m,n} = \frac{1}{2}(\mathrm{Min}_{0 \leqslant i \leqslant r} V_i^{m,n} + \mathrm{Max}_{0 \leqslant i \leqslant r} V_i^{m,n})$ and $w_i \geqslant 0$, then the error $R(t) - H^{m,n}(t)$ is simply bounded by $\frac{1}{2}(\mathrm{Max}_{0 \leqslant i \leqslant r} V_i^{m,n} - \mathrm{Min}_{0 \leqslant i \leqslant r} V_i^{m,n})\, B_m^{m+n}(t)$.

This paper investigates the conditions under which $h\langle m, n\rangle$ and $H\langle m, n\rangle$ converge to $R(t)$ as $\lim_{n \to \infty}(m/n) = \alpha(\geqslant 0)$. The two cases of greatest interest are $n = 0$ or $m = 0$ (Taylor interpolation) and $m = n$ (two-point Hermite interpolation).

Conventionally, such analyses proceed by examining the roots of $w(t)$, and convergence is assured if all roots lie outside a certain region in the complex plane. This sufficient convergence condition is derived in Section 2.

Section 3 develops the main results of the paper. Section 3.1 derives the relationship between polynomials $h^{m,n}(t)$ and $H^{m,n}(t)$ and gives the error terms for approximations $h\langle m, n\rangle$ and $H\langle m, n\rangle$. These error terms serve as the basis for the convergence conditions. Section 3.2 derives a recursive formula to compute the coefficients of $V^{m,n}(t)$ which is necessary to estimate the error terms for $h\langle m, n\rangle$ and $H\langle m, n\rangle$. Convergence conditions are obtained in Section 3.3 based on the preliminary results of Section 3.1 and Section 3.2. Section 3.4 provides concrete conditions for the low degree cases $r = 1$, 2, and 3, and a comparison is made between the condition in Section 2 and that presented in Section 3.3. Convergence for a more general case is discussed in Section 3.5. Finally, in Section 4 we apply these results to rational Bernstein–Bézier curves.

2. CONVERGENCE CONDITION FOR $h\langle s, s\rangle$ BY TRADITIONAL TECHNIQUES

Let $R(t)$ and $h^{s,s}(t)$ be defined as in (1) and (4), respectively. This section uses the classical approach to determining the convergence of $h\langle s, s\rangle$ as $s \to +\infty$.

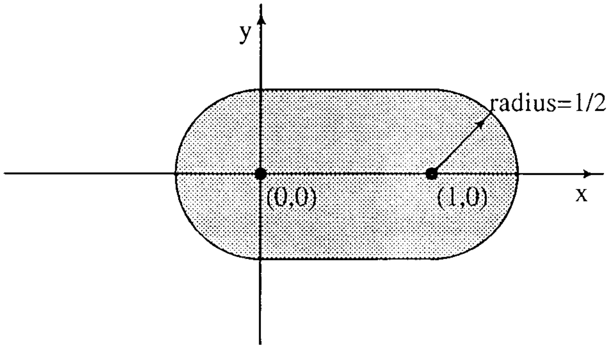

FIG. 1. Divergence region for Theorem 1.

THEOREM 1. *A sufficient condition for $h^{s,s}(t)$ to converge to $R(t)$ $(0 \leqslant t \leqslant 1)$ as $s \to +\infty$ is that none of the roots of $w(t)$ lie within a distance of $\frac{1}{2}$ of any point on the line segment $(0,0)$–$(1,0)$ in the complex plane (see Fig. 1).*

Proof. $R(t)$ can be written

$$R(t) = \frac{P(t)}{w(t)} = \sum_k \sum_l \frac{a_{kl}}{(t - z_k)^l}, \tag{9}$$

where $w(z_k) = 0$, a_{kl} are constants, and l is the multiplicity of (possibly complex) root z_k. By *the reminder theorem* for polynomial interpolation (see [2]), we have

$$R(t) - h^{s,s}(t) = \frac{R^{(2s)}(\theta)}{(2s)!} t^s (1-t)^s, \qquad 0 \leqslant \theta \leqslant 1. \tag{10}$$

Now we need to determine when

$$\lim_{s \to +\infty} (R(t) - h^{s,s}(t)) = 0, \qquad 0 \leqslant t \leqslant 1. \tag{11}$$

Simple calculation shows that

$$R^{(2s)}(t) = \sum_k \sum_l a_{kl} \frac{(l + 2s - 1)!}{(l-1)!} \frac{1}{(t - z_k)^{l+2s}}.$$

Thus we only need to determine, for any fixed k, l, when

$$L_{kl}(\theta, t)=\frac{(l+2s-1)!}{(2s)!\,(l-1)!}\left(\frac{t(1-t)}{(\theta-z_k)^2}\right)^s\frac{1}{(\theta-z_k)^l}$$
$$=\prod_{j=0}^{2s-1}\left(1+\frac{l-1}{j+1}\right)\left(\frac{t(1-t)}{(\theta-z_k)^2}\right)^s\frac{1}{(\theta-z_k)^l}\to 0, \qquad \text{as} \quad s\to+\infty.$$

Denote the distance between z_k and line segment $(0, 0)$–$(1, 0)$ by δ_k. Since

$$\prod_{j=0}^{2s-1}\left(1+\frac{l-1}{j+1}\right)=O(s^{l-1}),$$
$$\left|\frac{t(1-t)}{(\theta-z_k)^2}\right|^s\leqslant\left(\frac{1}{4\delta_k^2}\right)^s,$$
$$\left|\frac{1}{(\theta-z_k)^l}\right|\leqslant\frac{1}{\delta_k^l},$$

convergence will occur if

$$\delta_k>\tfrac{1}{2}. \tag{12}$$

Thus a sufficient condition for the convergence is that no real or complex root z_k of $Q(t)$ lies within a distance $\frac{1}{2}$ of the real interval $[0, 1]$. This completes the proof. ▮

In like manner, it can be shown that a sufficient condition for $h^{s,0}(t)$ or $h^{0,s}(t)$ converging to $R(t)$ $(0\leqslant t\leqslant 1)$ as $s\to+\infty$ is that none of the roots of $Q(t)$ lie within a distance 1 of the real interval $[0, 1]$.

3. CONVERGENCE CONDITION FOR HYBRID APPROXIMATION

In this section, we first point out the relationship between two types of polynomial approximation, $h\langle m, n\rangle$ and $H\langle m, n\rangle$, and then derive the approximation error terms for $h\langle m, n\rangle$ and $H\langle m, n\rangle$ and a recursive formula for computing $V_i^{m,n}$. Based on these preliminary results, convergence conditions are obtained and generalization is discussed.

3.1. *Remainder Terms for $h\langle m, n\rangle$ and $H\langle m, n\rangle$*

The ordinary Hermite interpolation approximation $h\langle m, n\rangle$ and hybrid approximation $H\langle m, n\rangle$ are closely related. Their relationship is expressed in the following theorem.

THEOREM 2. *Let $h^{m,n}(t)$ and $H^{m,n}(t)$ be defined as in* (4) *and* (8), *respectively*; *then*

$$h_i^{m,n} = \begin{cases} \dfrac{i}{m+n-1} H_{i-1}^{m-1,n-1} + \left(1 - \dfrac{i}{m+n-1}\right) H_i^{m-1,n-1}, \\ \qquad 0 \leqslant i \leqslant m+n-1, \quad i \neq m-1, m, \\ \dfrac{m-1}{m+n-1} H_{m-2}^{m-1,n-1} + \dfrac{n}{m+n-1} V_0^{m-1,n-1}, & i = m-1, \\ \dfrac{m}{m+n-1} V_r^{m-1,n-1} + \dfrac{n-1}{m+n-1} H_m^{m-1,n-1}, & i = m, \end{cases} \tag{13}$$

$$H_i^{m,n} = \frac{i}{m+n} h_{i-1}^{m,n} + \left(1 - \frac{i}{m+n}\right) h_i^{m,n}, \qquad 0 \leqslant i \leqslant m+n, \quad i \neq m. \tag{14}$$

That is, if we degree elevate polynomial $h^{m,n}(t)$, its coefficients and those of polynomial $H^{m,n}(t)$ differ only in the one coefficient $H_m^{m,n}$.

Proof. The proof of (14) is omitted since it is similar to that of (13). Noting that Bernstein polynomials are symmetric with respect to t and $1-t$, we only need to prove (13) for $i = 0, 1, ..., m-1$.

From

$$\tilde{H}^{m-1,n-1}(t) \equiv R(t)$$

we get

$$\frac{d^i h^{m,n}(0)}{dt^i} = \frac{d^i \tilde{H}^{m-1,n-1}(0)}{dt^i}, \qquad i = 0, 1, ..., m-1.$$

Expanding both sides of the above equation, we arrive at

$$\begin{aligned} &\frac{(m+n-1)!}{(m+n-1-i)!} \sum_{k=0}^{i} (-1)^k \binom{i}{k} h_{i-k}^{m,n} \\ &\quad = \frac{(m+n-2)!}{(m+n-2-i)!} \sum_{k=1}^{i} (-1)^k \binom{i}{k} H_{i-k}^{m-1,n-1} + \frac{(m+n-2)!}{(m+n-2-i)!} \\ &\qquad \times \begin{cases} H_i^{m-1,n-1}, & 0 \leqslant i \leqslant m-2, \\ V_0^{m-1,n-1}, & i = m-1. \end{cases} \end{aligned} \tag{15}$$

Using (15), the proof can be accomplished by mathematical induction on i. ∎

To get convergence conditions for the polynomial approximations $h\langle m, n\rangle$ and $H\langle m, n\rangle$, it is necessary to compute the error terms of the approximations. The following two theorems deal with that problem.

THEOREM 3. *The remainder term for $h\langle m, n\rangle$ is*

$$R(t) - h^{m,n}(t) = \frac{\hat{V}^{m-1,n-1}(t)}{w(t)} B_{m-1}^{m+n-2}(t) \tag{16}$$

which is bounded by

$$|R(t) - h^{m,n}(t)| \leqslant \frac{w_{\max}}{W_{\min}} \max_{\substack{0 \leqslant i \leqslant r \\ j=0,r}} |V_i^{m-1,n-1} - V_j^{m-1,n-1}|, \tag{17}$$

where $\hat{V}^{m-1,n-1}(t)$ is a degree $r+1$ Bernstein polynomial,

$$\hat{V}^{m-1,n-1}(t) = \sum_{i=0}^{r+1} \hat{V}_i^{m-1,n-1} B_i^{r+1}(t),$$

$$\hat{V}_i^{m-1,n-1} = \frac{i}{r+1} w_{i-1}(V_{i-1}^{m-1,n-1} - V_r^{m-1,n-1}) + \left(1 - \frac{i}{r+1}\right) \times w_i(V_i^{m-1,n-1} - V_0^{m-1,n-1}), \qquad i = 0, 1, ..., r+1, \tag{18}$$

and $w_{\max} = \mathrm{Max}_{0 \leqslant i \leqslant r} |w_i|$, $W_{\min} = \mathrm{Min}_{0 \leqslant t \leqslant 1} |w(t)|$.

Proof. Degree elevating $\tilde{H}^{m-1,n-1}(t)$ and using relation (14), we obtain

$$\tilde{H}^{m-1,n-1}(t) = \sum_{i=0,\, i \neq m-1,\, m}^{m+n-1} h_i^{m,n} B_i^{m+n-1}(t) + \left(\frac{m-1}{m+n-1} H_{m-2}^{m-1,n-1} + \frac{n}{m+n-1} V^{m-1,n-1}(t)\right) \times B_{m-1}^{m+n-1}(t) + \left(\frac{m}{m+n-1} V^{m-1,n-1}(t) + \frac{n-1}{m+n-1} H_m^{m-1,n-1}\right) B_m^{m+n-1}(t).$$

Thus

$$
\begin{aligned}
R(t) - h^{m,n}(t) &= \tilde{H}^{m-1,n-1}(t) - h^{m,n}(t) \\
&= \frac{n}{m+n-1} \left(V^{m-1,n-1}(t) - V_0^{m-1,n-1} \right) B_{m-1}^{m+n-1}(t) \\
&\quad + \frac{m}{m+n-1} \left(V^{m-1,n-1}(t) - V_r^{m-1,n-1} \right) B_m^{m+n-1}(t).
\end{aligned}
\tag{19}
$$

After simplifying (19), we get (16); (17) is simply from (16) and

$$
\begin{aligned}
\operatorname*{Max}_{0 \leqslant t \leqslant 1} |\hat{V}^{m-1,n-1}(t)| &\leqslant \operatorname*{Max}_{1 \leqslant i \leqslant r} |\hat{V}_i^{m-1,n-1}| \\
&\leqslant w_{\max} \operatorname*{Max}_{\substack{0 \leqslant i \leqslant r \\ j=0,\, r}} |V_i^{m-1,n-1} - V_j^{m-1,n-1}|. \quad \blacksquare
\end{aligned}
$$

THEOREM 4. *The remainder term for $H\langle m, n \rangle$ is*

$$
R(t) - H^{m,n}(t) = \left(V^{m,n}(t) - H_m^{m,n} \right) B_m^{m+n}(t), \tag{20}
$$

which is bounded by

$$
|R(t) - H^{m,n}(t)| \leqslant \frac{2w_{\max}}{W_{\min}} \operatorname*{Max}_{1 \leqslant i \leqslant r} |V_i^{m,n} - V_0^{m,n}|, \tag{21}
$$

where $w_{\max}$ and $W_{\min}$ are defined as in Theorem 3.

Proof. Equation (20) is directly from (6) and (8); (21) can be obtained by noting that $\operatorname{Min}_{0 \leqslant i \leqslant r} V_i^{m,n} \leqslant H_m^{m,n} \leqslant \operatorname{Max}_{0 \leqslant i \leqslant r} V_i^{m,n}$. $\blacksquare$

3.2. *Recursive Formula for $V_i^{m,n}$*

From Theorems 3 and 4, we see that the key to obtaining convergence conditions is to estimate $V_i^{m,n} - V_0^{m,n}$. This section derives the recursive formula for computing $V_i^{m,n} - V_0^{m,n}$.

THEOREM 5. *$V_i^{m,n} - V_0^{m,n}$ has the recursive formula*

$$
\begin{pmatrix} V_1^{m+1,n} - V_0^{m+1,n} \\ \vdots \\ V_r^{m+1,n} - V_0^{m+1,n} \end{pmatrix} = \frac{m+1}{m+n+1} W_r^{10} \begin{pmatrix} V_1^{m,n} - V_0^{m,n} \\ \vdots \\ V_r^{m,n} - V_0^{m,n} \end{pmatrix} \tag{22}
$$

$$\begin{pmatrix} V_1^{m,n+1} - V_0^{m,n+1} \\ \vdots \\ V_r^{m,n+1} - V_0^{m,n+1} \end{pmatrix} = \frac{n+1}{m+n+1} W_r^{01} \begin{pmatrix} V_1^{m,n} - V_0^{m,n} \\ \vdots \\ V_r^{m,n} - V_0^{m,n} \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} V_1^{m+1,n+1} - V_0^{m+1,n+1} \\ \vdots \\ V_r^{m+1,n+1} - V_0^{m+1,n+1} \end{pmatrix}$$

$$= \frac{2(m+1)(n+1)}{(m+n+2)(m+n+1)} W_r^{11} \begin{pmatrix} V_1^{m,n} - V_0^{m,n} \\ \vdots \\ V_r^{m,n} - V_0^{m,n} \end{pmatrix}, \quad m, n = 0, 1, 2, ..., \tag{24}$$

where

$$W_1^{10} = 1 - g_0,$$

$$W_r^{10} = \begin{pmatrix} 1-g_0 & g_1 & 0 & 0 & \cdots & 0 & 0 \\ -g_0 & 1 & g_2 & 0 & \cdots & 0 & 0 \\ -g_0 & 0 & 1 & g_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -g_0 & 0 & 0 & 0 & \cdots & 1 & g_{r-1} \\ -g_0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{25}$$

$$W_1^{01} = 1 - \frac{1}{g_0},$$

$$W_r^{01} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{g_{r-1}} \\ \frac{1}{g_0} & 1 & 0 & 0 & \cdots & 0 & -\frac{1}{g_{r-1}} \\ 0 & \frac{1}{g_1} & 1 & 0 & \cdots & 0 & -\frac{1}{g_{r-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{g_{r-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{g_{r-2}} & 1-\frac{1}{g_{r-1}} \end{pmatrix}, \tag{26}$$

$$W_1^{11} = \frac{1}{2}\left(2 - g_0 - \frac{1}{g_0}\right),$$

$$W_r^{11} = \frac{1}{2}\begin{pmatrix} 2-g_0 & g_1 & 0 & \cdots & 0 & 0 & -\frac{1}{g_0} \\ \frac{1}{g_1}-g_0 & 2 & g_2 & \cdots & 0 & 0 & -\frac{1}{g_1} \\ -g_0 & \frac{1}{g_2} & 2 & \cdots & 0 & 0 & -\frac{1}{g_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -g_0 & 0 & 0 & \cdots & 2 & g_{r-2} & -\frac{1}{g_{r-3}} \\ -g_0 & 0 & 0 & \cdots & \frac{1}{g_{r-2}} & 2 & g_{r-1}-\frac{1}{g_{r-2}} \\ -g_0 & 0 & 0 & \cdots & 0 & \frac{1}{g_{r-1}} & 2-\frac{1}{g_{r-1}} \end{pmatrix}, \tag{27}$$

$$g_i = \frac{(r-i)\, w_{i+1}}{(i+1)\, w_i}, \qquad i = 0, 1, ..., r-1. \tag{28}$$

Proof. We only prove (25), the others are similar.

First, we represent $V^{m,n}(t)$ as a hybrid polynomial of degree one,

$$V^{m,n}(t) = V_0^{m,n} B_0^1(t) + \bar{V}_0^1(t)\, B_1^1(t), \tag{29}$$

where $\bar{V}_0^1(t)$ is the variable coefficient.

By degree elevation, we have

$$\begin{aligned} \tilde{H}^{m,n}(t) = \sum_{i=0,\, i\neq m,\, m+1}^{m+n+1} & \left(\frac{i}{m+n+1} H_{i-1}^{m,n} \right. \\ & \left. + \left(1 - \frac{i}{m+n+1}\right) H_i^{m,n}\right) B_i^{m+n+1}(t) \\ & + \frac{m}{m+n+1} H_{m-1}^{m,n} B_m^{m+n+1}(t) + \frac{n}{m+n+1} H_{m+1}^{m,n} B_{m+1}^{m+n+1}(t) \\ & + \frac{n+1}{m+n+1} V_0^{m,n} B_m^{m+n+1}(t) + \frac{m+1}{m+n+1} \bar{V}_0^1(t)\, B_{m+1}^{m+n+1}(t). \end{aligned} \tag{30}$$

From the identity

$$\tilde{H}^{m,n}(t) \equiv \tilde{H}^{m+1,n}(t) = \sum_{i=0,\, i\neq m+1}^{m+n+1} H_i^{m+1,n} B_i^{m+n+1}(t) + V^{m+1,n}(t)\, B_{m+1}^{m+n+1}, \tag{31}$$

we get

$$\begin{aligned} V^{m+1,n} &= \frac{m+1}{m+n+1}\,\bar{V}_0^1(t) + \frac{n}{m+n+1}\,H_{m+1}^{m,n} \\ &= \frac{m+1}{m+n+1}\,\frac{V^{m,n}(t) - V_0^{m,n}(1-t)}{t} + \frac{n}{m+n+1}\,H_{m+1}^{m,n} \end{aligned} \tag{32}$$

and, hence,

$$\begin{aligned} V_i^{m+1,n} &= \frac{m+1}{m+n+1}\,(V_0^{m,n} + (V_i^{m,n} - V_0^{m,n}) + g_i(V_{i+1}^{m,n} - V_0^{m,n})) \\ &\quad + \frac{n}{m+n+1}\,H_{m+1}^{m,n} \qquad (i = 0, 1, ..., \gamma;\ g_\gamma = 0). \end{aligned} \tag{33}$$

Now (25) can be obtained from (32). ▮

3.3. *Convergence Conditions for $h\langle s, s\rangle$ and $H\langle s, s\rangle$*

In this section, we take special interest in the Hermite interpolation case $m = n = s$ and derive the convergence condition for $h\langle s, s\rangle$ and $H\langle s, s\rangle$.

From the remainder terms in Theorems 3 and 4, we know the convergence condition depends on the magnitude of $V_i^{s,s} - V_0^{s,s}$. If

$$\lim_{s\to\infty} (V_i^{s,s} - V_0^{s,s}) = 0, \qquad i = 1, 2, ..., r; \tag{34}$$

then $h^{s,s}(t)$ and $H^{s,s}(t)$ must converge to $R(t)$ uniformly on $[0, 1]$.

Using recursive formula (24), we now get the concrete condition for the convergence of $h\langle s, s\rangle$ and $H\langle s, s\rangle$.

THEOREM 6. *Let W_r^{11} be as defined in* (27) *and let $\lambda_1, \lambda_2, ..., \lambda_r$ be the eigenvalues* (*not necessarily distinct*) *of matrix W_r^{11}. Then*

$$|\lambda_i| < 2, \qquad i = 1, 2, ..., r, \tag{35}$$

is a sufficient condition for $h^{s,s}(t)$ and $H^{s,s}(t)$ converging to $R(t)$. Especially, for $r = 1$, the condition is also necessary.

Proof. By recursive formula (24) and noting that $\tilde{H}^{0,0}(t) \equiv V^{0,0}(t) \equiv R(t)$, we have

$$\begin{Bmatrix} V_1^{s,s} - V_0^{s,s} \\ \vdots \\ V_r^{s,s} - V_0^{s,s} \end{Bmatrix} = \frac{s!}{(2s-1)!!} (W_r^{11})^s \begin{Bmatrix} R_1 - R_0 \\ \vdots \\ R_r - R_0 \end{Bmatrix}. \tag{36}$$

Let $W_r^{11} = Q_r^{11} J_r^{11} (Q_r^{11})^{-1}$, where J_r^{11} is the Jordan canonical form [3, 6] of matrix W_r^{11} and Q_r^{11} is an $r \times r$ invertible matrix. By the *Stirling formula* [4], we know

$$\frac{s!}{(2s-1)!!} = \frac{2^s s!^2}{(2s)!} = O\left(\frac{\sqrt{s}}{2^s}\right). \tag{37}$$

Since

$$\lim_{s \to \infty} \frac{\sqrt{s}}{2^s} (J_r^{11})^s = 0 \tag{38}$$

if and only if $|\lambda_i| < 2$, $i = 1, 2, ..., r$, by (36), the sufficiency is proved.

In the case of $r = 1$, the eigenvalue of matrix W_1^{11} is $\lambda_1 = \frac{1}{2}(2 - g_0 - 1/g_0)$ and the remainder term for $h\langle s, s\rangle$ is

$$\begin{aligned} R(t) - h^{s,s}(t) &= \frac{(w_1 - w_0)\, B_1^2(t)}{2w(t)} \cdot (V_1^{s-1,s-1} - V_0^{s-1,s-1})\, B_{s-1}^{2s-2}(t) \\ &= \frac{(w_1 - w_0)(R_1 - R_0)\, B_1^2(t)}{2w(t)} \cdot \frac{\lambda_1^{s-1}(s-1)!}{(2s-3)!!} B_{s-1}^{2s-2}(t). \end{aligned} \tag{39}$$

We assume $(w_1 - w_0)(R_1 - R_0) \neq 0$ which means $R(t)$ is not a polynomial. Noting (37) and

$$\operatorname*{Max}_{0 \leqslant t \leqslant 1} B_{s-1}^{2s-2}(t) = O\left(\frac{1}{\sqrt{s}}\right),$$

we see $h^{s,s}(t)$ converges to $R(t)$ if and only if

$$|\lambda_1| < 2. \tag{40}$$

Thus the condition is also necessary.

Similarly, we can prove for $H\langle s, s\rangle$ that the condition is also necessary. ∎

By combining the result of the special case $r = 1$ and traditional techniques, we can obtain a much more interesting result.

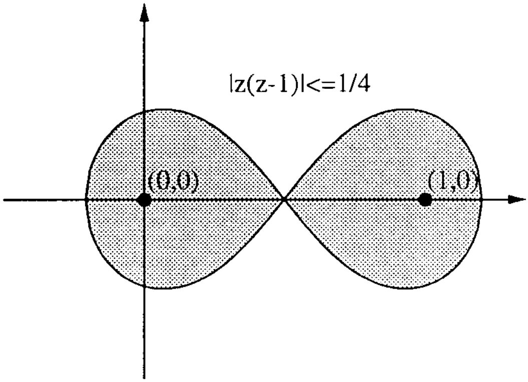

FIG. 2. Divergence region for Theorem 7.

THEOREM 7. *$h^{m,n}(t)$ and $H^{m,n}(t)$ converge to $R(t)$ if and only if all the roots z_i of polynomial $w(t)$ satisfy*

$$|z_i(1-z_i)| > 1/4; \tag{41}$$

i.e., the roots all lie on the outside of the region $D = \{z \in \mathscr{C} : |z(1-z)| \leqslant \frac{1}{4}\}$ as depicted in Fig. 2.

Proof. Let us first prove the simplest case, $r = 1$. In this case, $z_0 = w_0/(w_0 - w_1)$ and $W_1^{11} = 1/2z_0(1-z_0)$; (41) follows immediately from Theorem 6.

For the general case, we write $R(t)$ as in (9). If the roots of $w(t)$ are all simple (multiplicity 1), by (39), the remainder term for approximation $h\langle s, s\rangle$ is

$$R(t) - h^{s,s}(t) = \sum_k \frac{a_k}{\omega_k(t)} \left(\frac{1}{2z_k(1-z_k)}\right)^{s-1} \frac{(s-1)!}{(2s-3)!!} B_1^2(t)\, B_{s-1}^{2s-2}(t). \tag{42}$$

Thus $h^{s,s}(t) \to R(t)$ if and only if (41) holds.

If $w(t)$ contains multiple roots, the proof is much more tedious but still can be done by computing the error term and estimating it. The details are omitted. ∎

If D_0 denotes the shaded region in Fig. 1, it is easy to see $D \subset D_0$. Thus condition (41) relaxes the condition obtained by the traditional method.

3.4. *Convergence Criteria for $h\langle s, s\rangle$ and $H\langle s, s\rangle$ of degree* 1, 2, *and* 3

This section looks in detail at the convergence of $h\langle s, s\rangle$ and $H\langle s, s\rangle$ for rational Bernstein polynomials of degree $r = 1$, 2, and 3. It is shown that the sufficient condition from Section 2 is more restrictive than the condition in Section 3.3.

For $r=1$, from Section 3.3 it is seen that $h\langle s, s\rangle$ and $H\langle s, s\rangle$ uniformly converge to $R(t)$ if and only if

$$3-2\sqrt{2}<\frac{w_1}{w_0}<3+2\sqrt{2}. \tag{43}$$

Note that the condition obtained from Section 2 is

$$\frac{1}{3}<\frac{w_1}{w_0}<3. \tag{44}$$

Before considering the cases $r=2$ and 3, we need the following lemmas.

3.4.1. *Lemmas*

LEMMA 1. *The necessary and sufficient condition for all the roots of polynomial $f(t)$ satisfying*

$$|t|<c \qquad (c>0)$$

is that the real parts of all the roots of the polynomial

$$(1+u)^d f\left(\frac{1-u}{1+u}c\right)$$

are positive, where d is the degree of $f(t)$.

Proof. The roots of $f(t)$ satisfying $|t|<c$ is equivalent to the roots of $f(ct)$ satisfying $|t|<1$. Note that the transformation

$$u=\frac{1-t}{1+t} \qquad \text{or} \qquad t=\frac{1-u}{1+u}$$

maps the inside (outside) of the unit circle onto the positive (negative) half complex plane, and the lemma is proved. ∎

LEMMA 2. *The necessary and sufficient condition for the real parts of all roots of the polynomial*

$$p_d(u)=\sum_{i=0}^{d} p_{di}u^i, \qquad p_{dd}>0, \quad d=1,\ 2,\ 3$$

to be positive is that

- $d=1, 2$; $(-1)^{d-i} p_{di}>0$, $0\leqslant i\leqslant d-1$;
- $d=3$; $(-1)^{3-i} p_{3i}>0$, $i=0, 1, 2$, *and* $p_{30}p_{33}-p_{31}p_{32}>0$.

Proof. We prove the case $d=3$ only. Let u_1, u_2 and u_3 be the three roots of $p_3(u)$ and let

$$U_1=u_1+u_2+u_3, \qquad U_2=u_1u_2+u_2u_3+u_3u_1,$$
$$U_3=u_1u_2u_3, \qquad U_4=(u_1+u_2)(u_1+u_3)(u_2+u_3).$$

We claim the necessary and sufficient condition for $\mathrm{Re}(u_i)>0$, $i=1, 2, 3$, is

$$U_i>0, \qquad i=1, 2, 3, 4.$$

Since necessity is easy to prove, we only prove sufficiency.

If the roots of $p_3(u)$ are all real, from $U_i>0$, $i=1, 2, 3, 4$, we see u_i must be positive for $i=1, 2, 3$. If there exist a pair of complex roots, say $u_1=a+bi$, $u_2=a-bi$, and $u_3=c$, then

$$U_1=2a+c, \qquad U_2=a^2+b^2+2ac, \qquad U_3=(a^2+b^2)\,c,$$
$$U_4=2a((a+c)^2+b^2).$$

From $U_i>0$, $i=1, 2, 3, 4$, we obtain $a>0$ and $c>0$. This proves sufficiency.

Finally, since (see [1])

$$p_{3i}=(-1)^{3-i}\,U_{3-i}p_{33}\ (i=0, 1, 2), \qquad U_4=U_1U_2-U_3,$$

the lemma is confirmed. ∎

From Lemma 1 and Lemma 2, we immediately get

LEMMA 3. *Let u_{di}, $i=1, 2, ..., d$, be the roots of polynomial*

$$p_d(u)=\sum_{i=0}^{d} p_{di}u^i, \qquad p_{dd}=1, \quad d=1, 2, 3.$$

Then $|u_{di}|<2$, $i=1, 2, ..., d$, if and only if

- $d=1$, $|p_{10}|<2$;
- $d=2$, $|p_{20}|<4$, $|2p_{21}|<p_{20}+4$;

- $d=3,\ -4<p_{31}<12,\ |p_{30}+4p_{32}|<2p_{31}+8,$

$$|3p_{30}-4p_{32}|<24-2p_{31}, \qquad (4p_{32}-p_{30})\,p_{30}>16(p_{31}-4).$$

Proof. By Lemma 1, $|u_{di}|<2$, $i=1, 2, ..., d$, hold if and only if the real parts of all the roots of the polynomial

$$h_d(u)=\sum_{i=0}^{d} 2^i p_{di}(1-u)^i\,(1+u)^{d-i}$$

are positive. By Lemma 2, Lemma 3 is proved. ∎

3.4.2. *Convergence Condition for Degree 2 Case*

THEOREM 8. *Let $R(t)$ be a quadratic rational function as defined in* (1) *and let*

$$\begin{aligned}\xi&=w_1/w_0,\\ \eta&=w_1/w_2.\end{aligned} \tag{45}$$

Then $h\langle s, s\rangle$ and $H\langle s, s\rangle$ are uniformly convergent if

$$\left|(1-\xi)(1-\eta)+\frac{(\xi-\eta)^2}{4\xi\eta}\right|<4, \tag{46}$$

$$|2(\xi+\eta-2)|<4+(1-\xi)(1-\eta)+\frac{(\xi-\eta)^2}{4\xi\eta}; \tag{47}$$

i.e., $(\xi,\eta)\in\Omega$ in the ξ–η plane as depicted in Fig. 3. Moreover, the convergence condition in Theorem 1 is equivalent to requiring that (ξ, η) be in the shaded region Ω_0.

Proof. Since

$$W_2^{11}=\begin{pmatrix}1-\xi & \dfrac{1}{4}\left(\dfrac{1}{\eta}-\dfrac{1}{\xi}\right)\\ \eta-\xi & 1-\eta\end{pmatrix},$$

we can obtain the *characteristic equation* of the matrix W_2^{11},

$$|\lambda I-W_2^{11}|=\lambda^2-v\lambda+e=0,$$

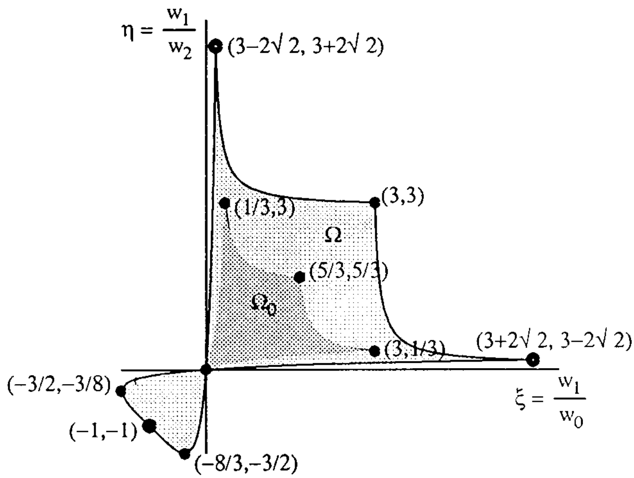

FIG. 3. Convergence region for degree two rational functions.

where

$$v = 2 - \xi - \eta,$$

$$e = (1-\xi)(1-\eta) + \frac{(\xi-\eta)^2}{4\xi\eta}.$$

The theorem follows from Lemma 3. ∎

Noting that $\Omega_0 \subset \Omega$, we see that the convergence condition from Theorem 1 is more restrictive than the one obtained by the new method.

3.4.3. *Convergence Condition for Degree* 3 *Case*

To get the convergence condition for cubic rational functions, we first compute the characteristic equation of the matrix W_3^{11},

$$|\lambda I - W_3^{11}| = \lambda^3 + a\lambda^2 + b\lambda + c = 0, \tag{48}$$

where

$$a = (-6w_0w_3 + 3w_0w_2 + 3w_1w_3)/2w_0w_3,$$

$$b = (9w_0w_3 - 12w_0w_2 + 3w_1w_0 - 12w_1w_3 + 9w_1w_2 + 3w_2w_3)/4w_0w_3,$$

$$c = (-6w_2w_3 + 6w_1w_3 - 18w_1w_2 - 2w_0w_3 + 6w_0w_2 + 9w_1^2 + w_0^2 + 9w_2^2 + w_3^2 - 6w_1w_0)/8w_0w_3. \tag{49}$$

From Lemma 3, we immediately arrive at

THEOREM 9. *Let $R(t)$ be a cubic rational function as defined in* (1). *Then $h^{s,s}(t)$ and $H^{s,s}(t)$ converge to $R(t)$ as $s \to \infty$ if*

$$-4 < b < 12, \qquad |4a + c| < 2b + 8, \qquad |4a - 3c| < 24 - 2b,$$
$$16(4 - b) > (c - 4a)\,c, \tag{50}$$

where a, b, c are defined in (49).

In the following, we give an example to demonstrate this.

EXAMPLE 1. Let $w_0 = 1$, $w_1 = 2$, $w_2 = 3$, $w_3 = 2$. It is easy to compute

$$a = \tfrac{9}{4}, \qquad b = \tfrac{3}{2}, \qquad c = \tfrac{1}{4},$$

and (50) holds. Thus the approximation is convergent.

3.5. *Convergence Condition for $h\langle s, 0\rangle$ and $H\langle s, 0\rangle$*

By taking a similar approach, we can get the convergence condition for approximations $h\langle s, 0\rangle$ (or $h\langle 0, s\rangle$) and $H\langle s, 0\rangle$ (or $H\langle 0, s\rangle$).

THEOREM 10. *Let λ_i^{jk}, $i = 1, 2, ..., r$ be the eigenvalues of matrix W_r^{jk} (defined in* (25) *and* (26)), *$jk = 10$ or 01. If $|\lambda_i^{10}| < 1$ ($|\lambda_i^{01}| < 1$), $i = 1, 2, ..., r$, then $h^{s,0}(t)(h^{0,s}(t))$ and $H^{s,0}(t)(H^{0,s}(t))$ converge to $R(t)$.*

Proof. Similar to the proof of Theorem 6. ∎

For low degree cases $r = 1$, 2, and 3, we can also get concrete conditions. The detailed discussion is omitted.

3.6. *General Case*

The above results obtained in the cases $m = n = s$ and $m = 0$ (or $n = 0$) can be extended to the more general case $\lim_{n \to +\infty}(m/n) = \alpha$ ($\alpha \geqslant 0$). We only illustrate the case $\alpha = 2$ briefly.

Proceeding as before, we can get the recursive formula

$$\begin{pmatrix} V_1^{a+2s,s} - V_0^{a+2s,s} \\ \vdots \\ V_r^{a+2s,s} - V_0^{a+2s,s} \end{pmatrix}$$
$$= \frac{(a+2s)!\, s!\, 3^s}{(a+3s)!} (W_r^{21})^s \begin{pmatrix} V_1^{a,0} - V_0^{a,0} \\ \vdots \\ V_r^{a,0} - V_0^{a,0} \end{pmatrix}, \qquad a, s = 0, 1, ..., \tag{51}$$

where

$$W_r^{21}=\frac{1}{3}\begin{pmatrix} 3-g_0(3-g_0+g_1) & g_1(3-g_0) & g_1g_2 & \cdots & 0 & 0 & -\frac{1}{g_0} \\ \frac{1}{g_1}-g_0(3-g_0+g_2) & 3-g_0g_1 & 3g_2 & \cdots & 0 & 0 & -\frac{1}{g_1} \\ -g_0(3-g_0+g_3) & \frac{1}{g_2}-g_0g_1 & 3 & \cdots & 0 & 0 & -\frac{1}{g_2} \\ -g_0(3-g_0+g_4) & -g_0g_1 & \frac{1}{g_3} & \cdots & 0 & 0 & -\frac{1}{g_3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -g_0(3-g_0+g_{r-3}) & -g_0g_1 & 0 & \cdots & 3g_{r-3} & g_{r-3}g_{r-2} & -\frac{1}{g_{r-4}} \\ -g_0(3-g_0+g_{r-2}) & -g_0g_1 & 0 & \cdots & 3 & 3g_{r-2} & g_{r-2}g_{r-1}-\frac{1}{g_{r-3}} \\ -g_0(3-g_0+g_{r-1}) & -g_0g_1 & 0 & \cdots & \frac{1}{g_{r-2}} & 3 & 3g_{r-1}-\frac{1}{g_{r-2}} \\ -g_0(3-g_0) & -g_0g_1 & 0 & \cdots & 0 & \frac{1}{g_{r-1}} & 3-\frac{1}{g_{r-1}} \end{pmatrix}. \tag{52}$$

Since

$$\frac{(a+2s)!\,s!}{(a+3s)!}\sim\frac{2^{a+1}}{3^{a+1/2}}\frac{\sqrt{\pi s}\,2^{2s}}{3^{3s}}\qquad\text{as}\quad s\to+\infty,$$

by Theorem 3, we obtain

THEOREM 11. *Let* λ_i^{21} $(i=1,2,...,r)$ *be the* eigenvalues *of the matrix* W_r^{21}. *If*

$$|\lambda_i^{21}|<\tfrac{9}{4},\qquad i=1,2,...,r, \tag{53}$$

then the approximations $h\langle a+2s,s\rangle$ *and* $H\langle a+2s,s\rangle$ *converge to* $R(t)$.

4. CONVERGENCE OF RATIONAL CURVES

A rational Bernstein–Beźier curve is defined by

$$\mathbf{R}(t)=\frac{\sum_{i=0}^r w_i\mathbf{R}_iB_i^r(t)}{\sum_{i=0}^r w_iB_i^r(t)},\qquad 0\leqslant t\leqslant 1, \tag{54}$$

where $\mathbf{R}_i \in \mathscr{R}^3$ are control points, and $B_i^r(t)$ and $w_i \geqslant 0$ are defined as in (1).

As in the function case, we can define two types of polynomial approximation of rational curves $\mathbf{h}\langle m, n\rangle$ and $\mathbf{H}\langle m, n\rangle$ as in (4) and (8), respectively. The only difference is that the coefficients are replaced by points in $\mathscr{R}^3$. The main results obtained in the previous sections are still valid for curves. Most importantly, we have

THEOREM 12. *Let W_r^{11}, and λ_i be the same as in Theorem 6. Then*

$$|\lambda_i| < 2, \qquad i = 1, 2, ..., r, \tag{55}$$

is a sufficient condition for $\mathbf{h}\langle s, s\rangle$ and $\mathbf{H}\langle s, s\rangle$ converging to $\mathbf{R}(t)$. Furthermore, for $r = 1$, the condition is necessary and sufficient. Also, if the r vectors $\{\mathbf{R}_i - \mathbf{R}_0\}_{i=1}^r$ are linearly independent, $r = 2, 3, ...,$ then the condition is necessary and sufficient.

Figure 4 illustrates the hybrid curves of a cubic rational curve which converge to the cubic curve while Fig. 5 demonstrates divergent hybrid curves.

One difference between functions and curves we should mention here is that the same curve can be reparametrized using a different parameter; i.e., it can have different functional representations. Thus the approximation for

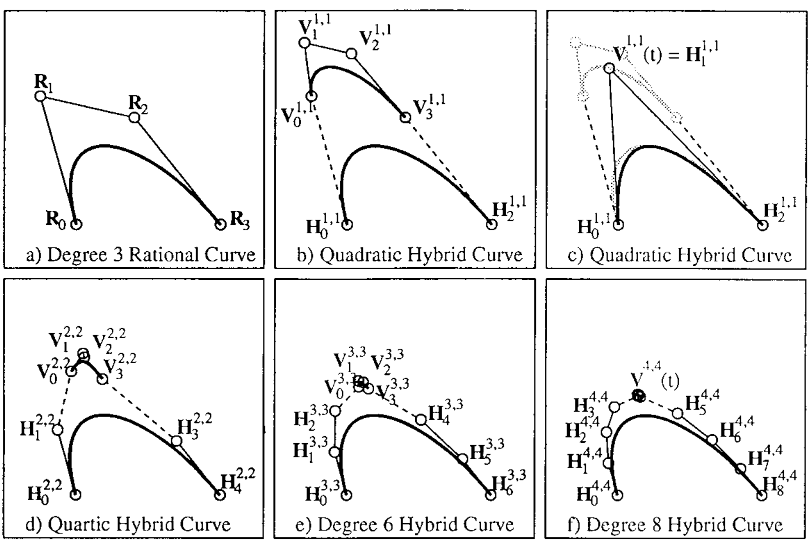

FIG. 4. Convergent hybrid curves.

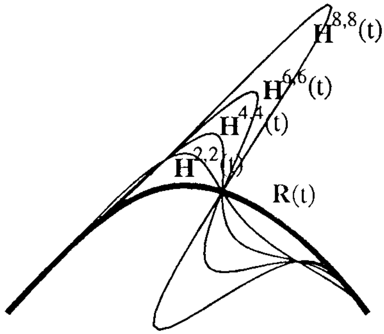

FIG. 5. Divergent hybrid curves.

curves can be a little different from that for functions. For example, any degree-1 rational curve can be reparameterized so that the polynomial approximation is always convergent. For quadratic curves we have the following result.

THEOREM 13. *Let* $\mathbf{R}(t)$ *be a quadratic rational Bernstein–Bézier curve as defined in* (54). *Let* ξ, η, *and* Ω *be as in Theorem* 8. *If* $(\xi, \eta) \notin \Omega$, *but* $\xi\eta = w_1^2/w_0 w_2 < 9$, *then* $\mathbf{R}(t)$ *can be reparameterized so that the approximations* $\mathbf{h}\langle s, s\rangle$ *and* $\mathbf{H}\langle s, s\rangle$ *are convergent.*

Proof. Obviously, there exists a point $(\xi_0, \eta_0) \in \Omega$ such that $\xi_0\eta_0 = w_1^2/w_0 w_2 < 9$. If we make the rational linear parameter transformation

$$t = \frac{\eta u}{\eta_0(1-u) + \eta u}, \tag{56}$$

then the curve $\mathbf{R}(t)$ becomes

$$\tilde{\mathbf{R}}(u) = \frac{\eta_0 \mathbf{R}_0 (1-u)^2 + \xi_0 \eta_0 \mathbf{R}_1 \cdot 2(1-u)\, u + \xi_0 \mathbf{R}_2 \cdot u^2}{\eta_0 (1-u)^2 + \xi_0 \eta_0 \cdot 2(1-u)\, u + \xi_0 \cdot u^2}, \qquad 0 \leqslant u \leqslant 1. \tag{57}$$

That means the curve itself is not changed, but it has a different parameter u and different weights $(\eta_0, \xi_0\eta_0, \xi_0)$. But since $(\xi_0, \eta_0) \in \Omega$, by Theorem 8, approximations $\mathbf{h}\langle s, s\rangle$ and $\mathbf{H}\langle s, s\rangle$ converge to $\tilde{\mathbf{R}}(u)$. This completes the proof. ∎

5. CONCLUSION

Based on the notion of hybrid polynomials, we have derived necessary and sufficient convergence criteria for various polynomial approximations of rational functions and rational curves. These conditions are better than those obtained using traditional methods.

Further study is needed to determine if the condition obtained in Theorem 6 is necessary and its relation to the necessary and sufficient condition obtained in Theorem 7. Also, experience suggests that convergence speed is related to eigenvalue magnitudes. This warrants closer study.

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REFERENCES

1. W. S. Burnside and A. W. Panton, "The Theory of Equations," Dover, New York, 1960.
2. P. J. Davis, "Interpolation and Approximation," Dover, New York, 1975.
3. C. P. Marshall, "Methods of Matrix Algebra," Academic Press, New York, 1965.
4. J. D. Nicholas, "Advanced Calculus with Applications," Macmillan, New York, 1982.
5. T. W. Sederberg and M. Kakimoto, Approximating rational curves using polynomial curves, *in* "NURBS for Curve and Surface Design" (G. Farin, Ed.), pp. 144–158, SIAM, Philadelphia, 1991.
6. H. F. Stephen, J. I. Arnold, and E. S. Lawrence, "Linear Algebra," Prentice–Hall, Englewood Cliffs, NJ, 1989.
7. G. Farin, "Curves and Surfaces for Computer Aided Geometric Design," Academic Press, Boston, 1988.